Spectral properties of a mixed system using an acoustical resonator

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We experimentally study the spectral properties of a mixed system using the flexural modes of a clover shaped plate. The system is called mixed because the corresponding ray dynamics has both chaotic and integrable regions in its phase space. The eigenvalue statistics show intermediate properties between the universal statistics corresponding to chaotic geometries which show Gaussian orthogonal ensemble statistics and integrable geometries that show Poisson statistics. We further investigate the Fourier transform of the peaks to study the influence of the length scales of the plate on the properties of the acoustic resonances. We observe a weak signal of the periodic orbits in the experimental data. Although some of the peaks in the Fourier transform of the eigenvalue spectrum correspond to the shortest stable periodic orbits, other strong peaks are also observed. To understand the role of symmetries, we start with a clover shaped plate belonging to the C_{4v} point symmetry group, and progressively reduce the symmetry by sanding one of the edges. A Shnirelman peak in $P(s)$ is observed for the highly symmetric situation due to level clustering.

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I. INTRODUCTION

The study of eigenvalues of quantum systems has shown the presence of universal features in the spectral statistics that are a direct manifestation of the corresponding classical dynamics [1,2]. The spectra of integrable systems are characterized by Poisson statistics, and completely chaotic systems show statistics described by one of the random matrix ensembles [3]. The intermediate case of ''mixed'' systems refer to the ones which have both integrable and chaotic regions in the corresponding classical phase space. Semiclassical formulas for mixed systems which assume an uncorrelated superposition of eigenvalues associated with different chaotic or regular regions in classical phase space were proposed [4]. However, numerical experiments showed that this formula is not always applicable [5], perhaps because of correlations present even in the chaotic levels [6]. Heuristic formulas [7] were observed to give better interpolations [8]. A universal theory which smoothly bridges the two limiting cases of fully integrable and fully chaotic classical dynamics has not yet emerged. The need for a better understanding of mixed systems arises from the fact that they are ubiquitous in nature. Furthermore, mixed systems with symmetries show new phenomena such as chaos assisted tunneling [6,9,10].

It is only recently being appreciated that the universality observed in quantum systems can be also applied to other wave systems. Experimental work with aluminum and quartz blocks established that the statistics of the vibrational modes follow the statistics of the eigenvalues of the Gaussian orthogonal ensemble (GOE) of random matrix theory [11–13]. An interesting issue is the relevance of periodic orbit theory for acoustical systems. However, analysis in terms of ray dynamics is complicated for blocks because the wave equation is vectorial. In addition, modes related to different symmetries are present, and mode conversion occurs at the boundaries. In contrast, it is far simpler to apply ideas from random matrix theory and periodic orbit theory to the vibrations of a plate. The reason for this is that the flexural and in-plane modes of a thin plate can be experimentally separated [14], and these modes are observed to be uncoupled [14,15]. The flexural modes at low frequencies are well described by a scalar biharmonic wave equation, and do not undergo mode conversion at the boundaries. Resonances with quality factors Q of up to 10^5 can be obtained at room temperature using fused quartz plates, thus giving significant eigenvalue statistics. Therefore the flexural modes of a plate are an interesting system to explore the spectral properties associated with mixed phase space dynamics due to the shape.

In this paper, we investigate the spectral statistics of the flexural modes of a clover-shaped [16] thin vibrating plate. The ray trajectories inside a clover geometry has both chaotic and integrable regions in its phase space. We perform measurements on a highly symmetric clover shaped plate (C_{4v} point symmetry group) to understand the role of symmetries and the length scales of the plate on the properties of the eigenvalues. We also sand one of the edges of the plate to study the effect on the spectral properties. A semiclassical theory of flexural modes of a plate [17] proposes that one expects a close correspondence between the resonances and ray periodic orbits for the acoustic systems, because the statistical properties of the spectrum are the same as that for the quantum billiard case. We study the Fourier transform of eigenvalues to test this idea. We find that some of the peaks may be associated with the main stable periodic orbit, but others did not correspond to stable periodic orbits. We also analyze the statistical properties of the acoustical resonances using the spectral spacing distribution $P(s)$ and the spectral rigidity $\Delta_3(L)$. We find that the eigenvalue statistics show intermediate properties between the universal GOE and Poisson statistics which depends on the symmetry of the system. A conjecture [18] based on Shnirelman's theorem states that a narrow (Shnirelman) peak in the distribution of nearest neighbor eigenvalue spacing is expected not only for nearly integrable systems but also for chaotic systems with a discrete symmetry, provided that the states with opposite sym-

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metry are separated in phase space. In the case of the highly symmetric clover shaped plate, a Shnirelman peak is observed in the first bin of $P(s)$, thus leading the distribution to deviate from Poisson. The $\Delta_3(L)$ at low L is observed to be higher than $L/15$ appropriate for a Poisson distribution. An intermediate behavior is observed in both $P(s)$ and $\Delta_3(L)$ after the initial high symmetry of the plate is completely broken.

II. CLOVER SHAPED PLATE

The geometry of the clover shaped plate used in our experiments is shown in Fig. 1. The geometry is similar to an equipotential contour of the quartic oscillator which is often used to study mixed systems [6,9]. The dimensions that control the geometry are the radius of the convex side, r; the radius of the concave sides, R; the distance between the longitudinal concave sides, $2X_m$; and the distance between the vertical concave sides, $2Y_m$ (see Fig. 1). The ray dynamics inside this shape was investigated numerically in detail, and reported in Ref. [16]. Here we summarize the most important properties pertaining to this paper. The mixed nature of the phase space is illustrated by the Birkhoff-Poincare section shown in Fig. 2(a). Because of the symmetries present in the geometry, it is sufficient to plot the phase space from $s=0$ to $s=s_{max}/4$, where s is the curvilinear abscissa along the boundary, and s_{max} is the perimeter of the plate boundary. Two main regular islands in the phase space are observed that correspond to trajectories which are confined to the vertical and longitudinal focusing areas. The regular regions occupy only about 6.3% of the total area, and therefore the system is mainly chaotic. Trajectories that hit only the concave sides are stable and belong to integrable regions, and trajectories that hit the convex sides are generally chaotic. The main stable periodic orbits that may be expected to influence the eigenvalue spectrum are shown in Figs. 2(b)–2(e).

Next we briefly discuss the wave equation and boundary conditions that govern the flexural modes of a thin plate. According to the Kirchhoff-Love model, the displacement $W(x,y)$ of the flexural modes is perpendicular to the plane of the plate, and obey the time-independent wave equation [19]

$$(\nabla^2-k^2)(\nabla^2+k^2)W(x,y)=0, \tag{1}$$

where k denotes the wave number, and x and y are the Cartesian coordinates. This equation is a good approximation provided the wavelength is much greater than the thickness of the plate h. The modes are two dimensional provided the wavelength is greater than twice the thickness. This implies that for a fused quartz plate with thickness $h=1.5875$ mm, the modes are two dimensional below 1.18 MHz.

In the case of a freely vibrating plate, the boundary conditions in the Kirchhoff-Love model are given by [19]

$$\frac{\partial^2 W(x,y)}{\partial x^2}+\nu\frac{\partial^2 W(x,y)}{\partial y^2}=0, \tag{2}$$

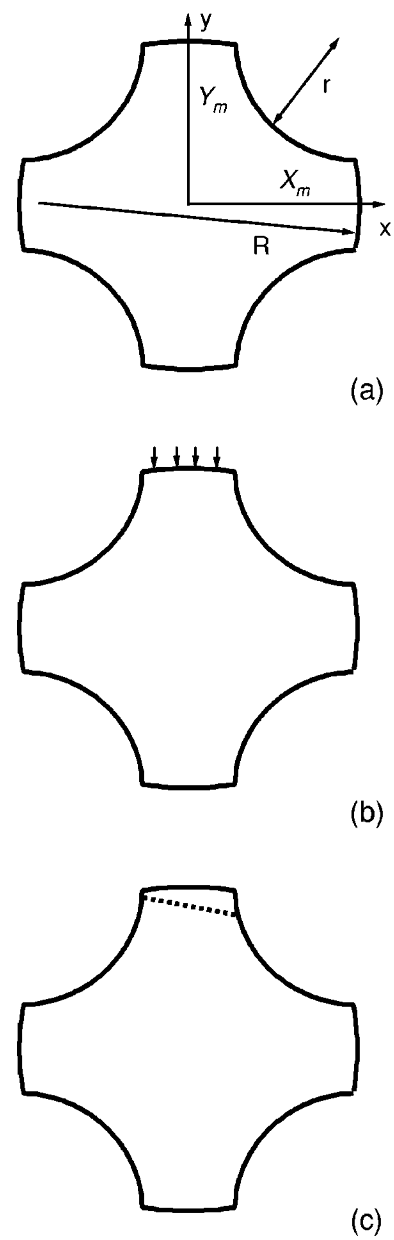

FIG. 1. (a) The clover geometry belongs to the C_{4v} point symmetry group ($X_m=Y_m=5.080$ cm, $r=3.556$ cm, and $R=9.144$ cm.) (b) The clover shaped plate with reflection symmetry about the vertical axis is obtained by sanding the edge of the plate as indicated. This shape has two symmetry classes of modes. (c) The asymmetric clover shaped plate is obtained by sanding the edge, as indicated by the dashed line.

$$\frac{\partial^3 W(x,y)}{\partial x^3}+(2-\nu)\frac{\partial^3 W(x,y)}{\partial x\partial y^2}=0. \tag{3}$$

The solutions differ from that of the Schrödinger wave equation for a billiard shape because of the coupled nature of the two boundary conditions. Another important difference is the edge modes that arise due to presence of the operator (∇^2-k^2) in Eq. (1). However, these modes decay exponentially from the edge of the plate, and therefore exist in a width $1/k$ near the boundary. The boundary modes are estimated to be

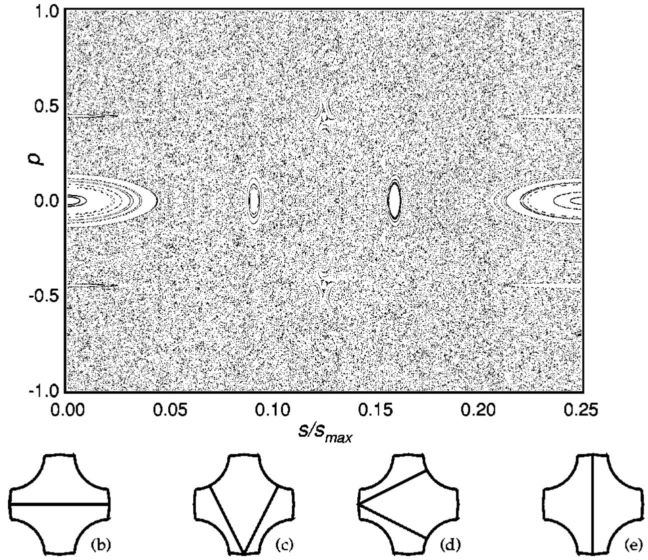

FIG. 2. (a) The Poincaré surface of section for the clover shaped plate shown in Fig. 1(a) obtained by launching rays in 1000 random directions. The random points indicate chaotic regions, and points along arcs indicate regular tori regions around stable orbits. The main stable periodic orbits corresponding to the center of the tori in the phase space along $p=0$ are shown in (b)–(e).

less than 5% of the modes, and do not appear to alter the universality of the eigenvalues [14,15].

The dispersion relation that relates the wave number k to the frequency f was accurately calculated for finite plates in Ref. [15], and is given by

$$k=\frac{12^{1/4}}{h\sqrt{\kappa}}\sqrt{\Omega}(1+a_1\Omega+a_2\Omega^2), \tag{4}$$

where $\kappa=\sqrt{2/(1-\nu)}$, $\Omega=2\pi fh/c_s$, and ν is the Poisson ratio. The factors a_1 and a_2 are functions of ν,

$$a_1=\frac{\sqrt{6}(17-7\nu)}{240\sqrt{1-\nu}},\quad a_2=\frac{607\nu^2+1726\nu-1353}{134\,400(1-\nu)}, \tag{5}$$

and were also calculated in Ref. [15]. For fused quartz, $\nu=0.16$ [20], and therefore $a_1=0.177$ and $a_2=-9.40\times10^{-3}$. Until recently only the first term in the expansion given in Eq. (4) was used to calculate k. However, we find that the correction terms are necessary to relate the peaks in the Fourier transform to the periodic orbits.

III. EXPERIMENTAL METHOD

The experimental setup is the same as described in previous experiments reported in Ref. [14]. The plate was precision machined by Insaco, Inc. to the dimensions shown in Fig. 1 to within 3 μm. The plate is kept on three piezoelectric transducers (one transmitter and two receivers) and the vibrations of the plate are measured using a HP4395A Network Analyzer. Therefore, the plate can vibrate freely inside the chamber, the only contact being made through the tiny ruby spheres which are attached to the transducers. The contacts reduce the Q but do not perturb the resonances.

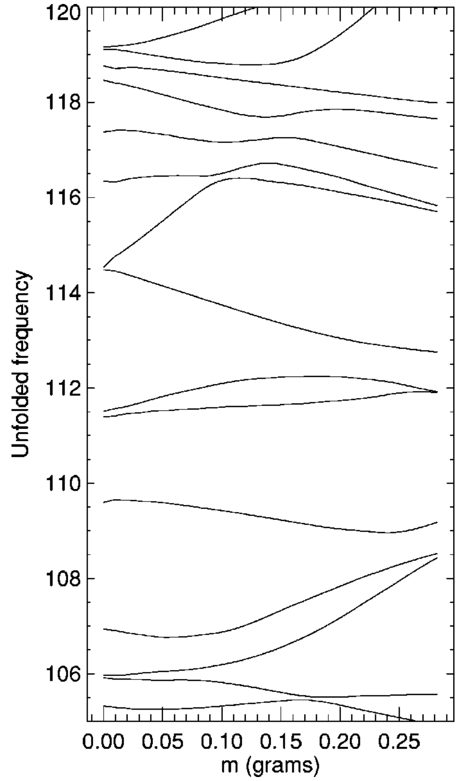

FIG. 3. Unfolded resonances of the clover shaped plate as a function of the removed mass m. Note the degeneracies at $m=0$ that are broken when the symmetry of the plate is destroyed by sanding the edge of the plate.

The plate and the transducers are placed inside a temperature controlled chamber at 300 K and at a pressure below 10^{-1} Torr to prevent losses due to air damping. To experimentally isolate the flexural class of modes [14], we increase the pressure inside the chamber, and measure the resonance spectrum at $\sim$300 Torr. The Q factor of the flexural modes decreases, whereas the Q factor for the extensional modes is unchanged. Thus the resonances corresponding to the flexural modes are identified. The flexural and extensional classes of modes have been shown to be uncoupled in a plate with a reflection symmetry through the midplane [15]. Avoided crossings are not observed between flexural and extensional modes in our experiments, consistent with earlier findings.

The transmission amplitude of the clover shaped plate with C_{4v} point group symmetry shown in Fig. 1(a) was measured in a low frequency interval between 52 and 352 kHz, and a high frequency interval between 800 and 1000 kHz, with a frequency resolution of 5/8 Hz. We then sanded off material at one of the edges as indicated in Fig. 1(b) in 61 small steps, the clover shaped plate having a reflection symmetry about the vertical axis. (Y_m was reduced by 1.75 %.) The evolution of the resonances was followed in the lower frequency interval, and is shown in Fig. 3. A number of mode splittings can be observed as the material is sanded off. Using this technique most of the near and exact degeneracies associated with the symmetries of the clover shaped plate

TABLE I. Comparison of the flexural modes measured experimentally for two frequency intervals and shapes of the plate, and the theoretical estimate using the Weyl formula.

Clover	Frequency interval (kHz)	Number of modes (experiment)	Number of modes (theory)	Percentage error
C_{4v}	52–352	506	525	3.6
C_{4v}	700–1000	875	963	9.1
asymmetric	52–352	489	511	4.3
asymmetric	800–1000	650	649	0.1

were accurately identified. The presence of large number of degeneracies in the unperturbed plate indicates that any error in the machining of the initial plate is negligible.

Finally, the remaining symmetry of the plate was also broken, as indicated in Fig. 1(c). Therefore acoustical spectra corresponding to shapes with three different point group symmetries were studied. In Table I, a comparison is made between the number of modes found experimentally at different frequency intervals, and the theoretical estimates using the Weyl formula derived in Ref. [15]. It can be seen from the table that fewer modes are counted in the case of the clover shaped plate with C_{4v} point group symmetry, due to the high number of near degeneracies present in the system compared to the case with no symmetries. However, it must be also noted that there are errors associated with the theoretical estimates because elastic constants are known only to within 1%.

IV. PROPERTIES OF THE FLEXURAL RESONANCES

Next we analyze the obtained eigenvalues using statistical measures to compare the data to the universal limits, and study the source of the deviations using Fourier transforms.

A. Spectral statistics

We obtained the distribution of nearest neighbor spacings $P(s)$ after unfolding the spectrum [2], where s is the difference between the nearest neighbor eigenvalues normalized by the mean level spacing. The data for the highly symmetrical clover shaped plate, the clover shaped plate with one reflection symmetry along the vertical axis, and the asymmetrical clover shaped plate shown in Fig. 4 and correspond to the eigenvalues between 52 and 352 kHz. (Similar distributions are also observed at high frequencies.) $P(s)$ for the clover shaped plate with C_{4v} point group symmetry deviates strongly from Poisson distribution, and a peak is observed in the first bin, indicating level clustering. As the symmetry is reduced, the peak in $P(s)$ disappears. For the clover shaped plate with no symmetries, $P(s)$ approaches the Wigner-Dyson distribution corresponding to GOE statistics.

The degeneracies in the eigenvalues of the highly symmetrical clover shaped plate occur because of the special symmetries present in the geometry. The symmetry group of the clover shaped plate shown in Fig. 1(a) is C_{4v} [21]. C_{4v} has five irreducible representations, one of which is doubly degenerate [22]. For shapes with this symmetry, the four nondegenerate representations each contribute one eighth of the number of modes, and the doubly degenerate representation contributes to the remaining half. The degeneracies are lifted when this high symmetry is slightly broken, for example by reducing one of the dimensions as in Fig. 1(b). This is accomplished in experiments by sanding the corresponding edge. Since the ''super-Poisson'' behavior in $P(s)$ occurs due the spatial symmetry of the clover shaped plate, the observed deviation at small s is a Shnirelman peak [18,23].

$P(s)$, excluding the first bin, was fitted with a scaled Poisson distribution following Ref. [18]. The scaled Poisson distribution is given by

$$P(s) = (1-\alpha)^2 e^{-(1-\alpha)s}, \tag{6}$$

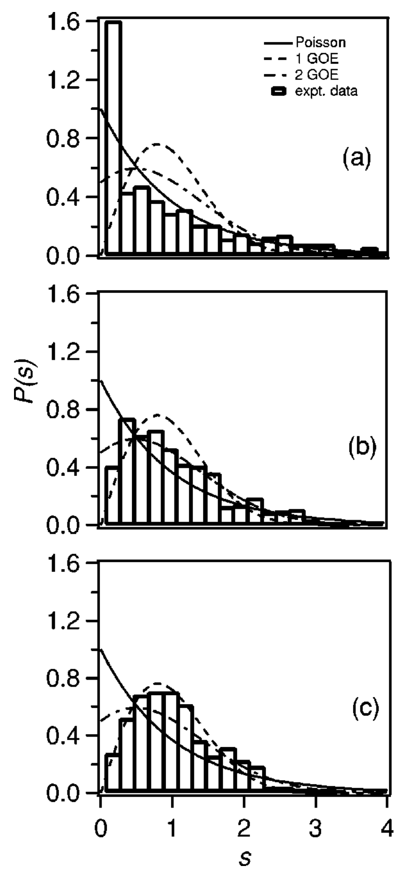

FIG. 4. The nearest-neighbor eigenvalue distribution $P(s)$ for (a) the clover shaped plate with C_4v symmetry, (b) the clover shaped plate with reflection symmetry, and (c) the asymmetric clover shaped plate. The first bin in (a) is well above the Poisson distribution because of the presence of degeneracies.

where the parameter α corresponds to the fraction of degenerate levels present. We obtain $\alpha=0.21$ for the clover shaped plate with C_{4v} symmetry shown in Fig. 4(a). Since half the modes are doubly degenerate, we expect $\alpha=0.25$. This value estimated on symmetry arguments is close to the α obtained by fitting the experimental data. The difference is possibly due to some degeneracies being unresolved even after sanding the edge of the plate. Although the initial prediction by Shnirelman referred to a nearly integrable system, we find a Shnirelman peak in a mixed system. This suggests that the Shnirelman effect can be extended to mixed systems, as proposed in Ref. [18].

$P(s)$ for the clover shaped plate with one reflection symmetry along the vertical axis [Fig. 1(b)] is shown in Fig. 4(b). The distribution is close to the two GOE curves obtained by mixing two independent GOE spectra [2], but deviates significantly for small spacings. The symmetry group of this clover shaped plate is C_s. C_s has two irreducible nondegenerate representations classifiable by parity with regard to the vertical symmetry plane. For shapes with this symmetry, the two nondegenerate representations each contribute one half of the number of modes. For a completely chaotic geometry this situation would be compared with the two GOE curves. We completely break the symmetry of the plate, as indicated in Fig. 1(c), to directly check that the deviations are due to the mixed dynamics in the plate, and to reduce the complications introduced due to mixing of two independent classes of modes. $P(s)$ is plotted in Fig. 4(c), and is observed to be close to the universal GOE distribution.

To obtain a quantitative measure of the distribution, we fit $P(s)$ with the Brody parameter β using [7]

$$P(s)=As^{\beta}e^{Bs^{\beta+1}}, \tag{7}$$

where A and B are normalization constants. The distribution has been used to characterize a distribution between the Poisson distribution which corresponds to $\beta=0$ and a GOE distribution corresponding to $\beta=1$. We obtain $\beta=0.80\pm0.05$ for the desymmetrized clover shaped plate at low frequencies, thus showing deviations from the GOE distribution. A similar $\beta=0.85\pm0.05$ was obtained using the data in the high frequency range from 800 to 1000 kHz. Therefore the details of the nature of the phase space has an impact on the spectral statistics [24].

We used the spectral rigidity $\Delta_3(L)$ to study the long range correlations in the eigenvalues for the three shapes (see Fig. 5). The definition of $\Delta_3(L)$ can be found along with the theoretical curves in Ref. [25], and L is the length of the interval over which the correlation is calculated. The $\Delta_3(L)$ curve for the clover shaped plate with C_{4v} symmetry shown in Fig. 5(a) lies above the Poisson distribution for $1<L<10$, possibly reflecting the fact that there are many degeneracies in the spectra. The $\Delta_3(L)$ for $L>10$ starts to saturate and lies between the Poisson and GOE distributions. The $\Delta_3(L)$ for the clover shaped plate with vertical reflection symmetry is shown in Fig. 5(b), and is observed to agree with the two GOE curves for $L<7$. However, the $\Delta_3(L)$ for $L>7$ increases above the two GOE curve possibly due to the mixed nature of the phase space. Finally, the $\Delta_3(L)$ for the desymmetrized case is shown in Fig. 5(c). The data are observed to follow the GOE curve for $L<7$, and then increases roughly linearly. Therefore the data clearly shows the effect of mixed nature of the phase space on the statistical properties of the flexural modes of the plate because good agreement with the GOE curve is observed, at least up to $L<25$ with chaotic shaped plates [14].

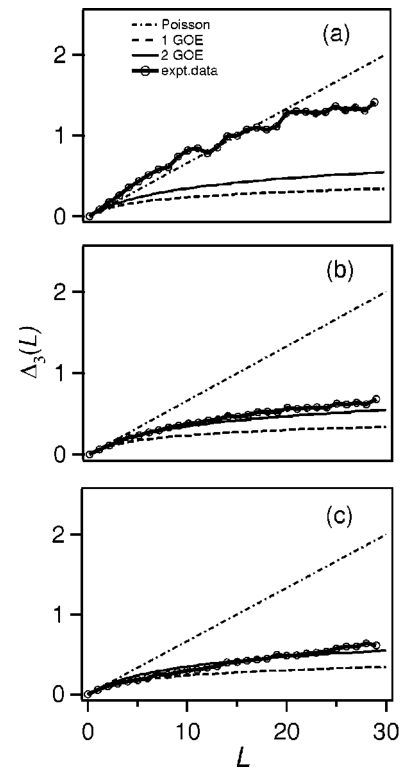

FIG. 5. The spectral rigidity $\Delta_3(L)$ of the eigenvalues of (a) the clover shaped plate with C_4v symmetry, (b) the clover shaped plate with reflection symmetry, and (c) the asymmetric clover shaped plate. The deviation of the experimental data for $L>7$ for the asymmetric clover shaped plate indicates the influence of the mixed phase space dynamics on the eigenvalues. The deviations in the other two cases are due to a combination of symmetry mixing and the mixed phase space dynamics.

B. Fourier transforms

To understand the nature of the deviations present in the spectra statistics, we calculated the square of the Fourier transforms using the formula

$$|F(l)|^2=n+2\sum_{i>j,1\leq i,j\leq n}\cos[(k_j-k_i)l], \tag{8}$$

where n represents the total number of resonances in a spectrum and k_i denotes the wave number of the ith eigenvalue. k_i were calculated from Eq. (4) using the resonances obtained experimentally. $|F(l)|^2$, corresponding to the clover shaped plate with C_{4v} symmetry in the frequency range from 52 to 352 kHz, is shown in Fig. 6(a), and that from 800 to 1000 kHz is shown in Fig. 6(b). A number of strong peaks are present in the experimental data. According to theoretical

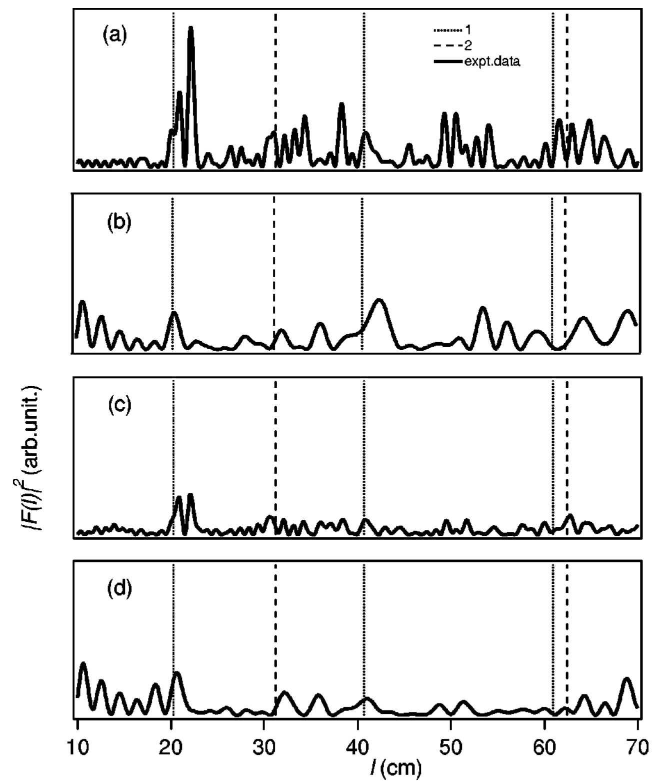

FIG. 6. $|F(l)|^2$ of the flexural modes for (a) the clover shaped plate with C_4v symmetry (52–352 kHz), (b) the clover shaped plate with C_4v symmetry (800–1000 kHz), (c) the clover shaped plate with reflection symmetry (52–352 kHz), and (d) the clover shaped plate with reflection symmetry (800–1000 kHz). The position of the main stable periodic orbits of the system is also shown in the graphs: 1 corresponds to the periodic orbits labeled (b) and (e) in Fig. 2, and 2 refers to the periodic orbits labeled (c) and (d).

expectations, $|F(l)|^2$ should show strong peaks at values corresponding to the length of the stable classical periodic orbits. The length corresponding to the shortest stable periodic orbits plotted in Fig. 2 are also indicated. A peak is observed near the shortest stable periodic orbit in both the high and low frequency data. However, other strong peaks are observed that cannot be directly assigned to the shortest stable periodic orbits.

$|F(l)|^2$ for the desymmetrized Clover for low and high frequencies are shown in Figs. 6(c) and 6(d), respectively. We note that desymmetrization leaves one set of stable periodic orbits intact. The peak corresponding to the shortest stable orbit is still observed roughly at the same position, while the location of other peaks is observed to change.

We note that this weak correspondence between the robust peak and the shortest periodic orbit is observed only when the higher order corrections for the wave numbers are included which were recently calculated in Ref. [15]. No peaks were observed to correspond to periodic orbits when only the first term in the dispersion relation shown in Eq. (4) was used to calculate k.

V. DISCUSSION

We have reported the spectral properties of a clover shaped thin vibrating plate which is an example of a mixed system. The eigenvalue statistics show intermediate properties between the universal statistics corresponding to chaotic shapes which show GOE statistics and integrable shapes that show Poisson statistics. The intermediate nature of the statistics was illustrated using the Brody parameter β.

This study also provides evidence of the possible relevance of the periodic orbits in acoustic systems in the Fourier transform of the eigenvalues. There is a strong peak at low and high frequencies for the clover shaped plate which is located near the shortest stable periodic orbit, but there are also others peaks which are observed. Bogomolny and Hugues developed a semiclassical theory of flexural vibrations of plates, and a trace formula for the density of states was calculated [17]. They found that the main difference between this formula and the Gutzwiller trace formula is the presence of a phase factor, because of the reflection of the waves from the boundary. Therefore, we expect that for acoustic systems there is also a close correspondence between resonances and classical periodic orbits. Hence the weak signal of the periodic orbits in the experimental data is puzzling.

In a parallel numerical study of clover shaped geometry, eigenvalues were calculated using the biharmonic equation with clamped boundary conditions [16]. Similar distributions for $P(s)$ and $\Delta_3(L)$ were obtained using the first 281 eigenvalues. Furthermore, $|F(l)|^2$ show peaks near stable and unstable periodic orbits. Surprisingly, peaks near unstable orbits are often stronger than those near stable orbits. Thus the statistical properties of the eigenvalues appear to be independent of the free boundary conditions used in the experiments, and depend only on the shape of the boundary.

We also experimentally investigated the role of the symmetry on the eigenvalues of plates. The effect of spatial symmetries of the clover shaped plate lead to the appearance of a Shnirelman peak in $P(s)$ even though the system is mixed. Thus the experimental data support the conjecture of Chirikov and Shepelyansky [18] that a Shnirelman peak can be expected not only for nearly integrable systems as first proposed, but also for completely integrable and chaotic systems.

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